

Four-photon entanglement from two-crystal geometry

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In this paper, a four-photon entangled state obtained directly from the down-conversion of a two-crystal geometry is analyzed. These two identically cut type-I crystals are oriented with their optic axes aligned in perpendicular planes. The applications of this kind of four-photon entangled state are also discussed.

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Entangled states of two or more particles not only play a central role in discussions of nonlocal quantum correlations [1], but they also form the basis of quantum information, including quantum cryptography [2], dense coding [3], teleportation [4], and quantum computation [5]. Particularly, an entangled state of more than two particles can be used for improved tests of local hidden variable theory, and is also a key ingredient for multiparty quantum communication [6]. To date, parametric down-conversion has proven to be the best method to generate entangled photon pairs [7]. Now it has become a routine to generate two-photon entangled states in the laboratory. Interference of photons generated by independent down-conversion processes enabled the first demonstration of a three-photon Greenberger-Horne-Zeilinger (GHZ) state [8] and even the four-photon GHZ state [9]. Quite recently, a kind of four-photon entangled state was observed directly in the type-II parametric down-conversion process [10].

In this paper, we show that a kind of four-photon entangled state can be directly generated by post-selection from a degenerate type-I two-crystal geometry, which has been used to generate a high-intensity entangled state of two photons [11,12]. In contrast to previous techniques, the generation of this four-photon entangled state does not require fragile interferometric setups, and also the intensity should be very high because of the type-I collinear parametric interaction. The application of this four-photon entangled state on quantum secret sharing is also introduced.

It is well known that a two-photon state can be generated via spontaneous parametric down-conversion. For a strong pulsed pump, there is a reasonable possibility of simultaneously producing four photons (two down-converted photon pairs) in one single pulse [10,14]. Consider the experimental setup of Fig. 1. A strong pulsed laser is used to pump two identically cut type-I crystals with their optic axes aligned in mutually perpendicular planes. By three-nonpolarization beam splitter and four-coincidence detection, one can post-select the four-photon state in one single pump pulse. The four-photon state from this parametric down-conversion process consists of three parts: the first one (part A) is two down-conversion photon pairs both coming from the first crystal with horizontal polarization, the second one (part B) is two pairs both coming from the second crystal with vertical polarization, and the third one (part C) is one pair coming from the first crystal with horizontal polarization and the other one coming from the second crystal with ver-

tical polarization. To get a four-photon polarization entangled state, first, photon pairs which come from the first crystal should be indistinguishable from those coming from the second crystal; secondly, the two pairs should be indistinguishable.

By adjusting the polarization of the pump to 45° and using a delay line [11,12], down-conversion photons from the first crystal and from the second crystal can be made indistinguishable. To ensure the temporal indistinguishability of two-photon pairs, narrowband filters are needed to make the coherence time of the down-converted four photons longer than the pump pulse duration [13]. For good overlapping of the spatial mode, it can be obtained by a single-mode fiber [10]. At this point, the four-photon state after the two crystals can be written as a coherent superposition of parts A, B, and C. The relative phase between them can be changed by controlling the relative phase between the horizontal and vertical components of the pump light. To decide the coefficients of the three parts (see the Appendix), we assume the probability of creating one photon pair in a single pulse is P_2 . For part C, the probability of simultaneously creating one pair from each crystal is P_2^2 (because the creation of the two pairs is independent). For part A (part B), the probability of simultaneously creating two pairs from one crystal is also P_2^2 due to the stimulation emission process [15,16]. Now the simplified version of the four-photon state can be written as

$$|\Psi^{(4)}\rangle = |4H_a\rangle + |4V_a\rangle + |2H_a, 2V_a\rangle. \quad (1)$$

Here $4H_a$ means four horizontal polarization photons in mode a , etc. The four-photon state can also be written as

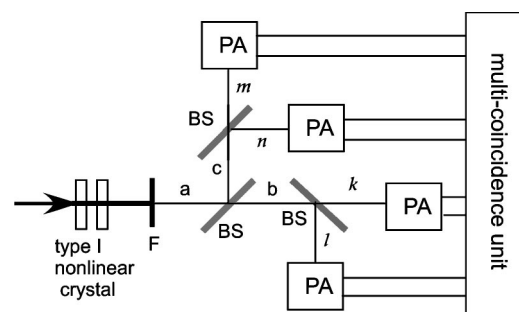


FIG. 1. Experimental setup to obtain the four-photon entanglement from a two-crystal geometry. F, pump filter; BS, 50-50 non-polarization beam splitter; PA, polarization analysis.

$$[(\hat{a}_H^\dagger)^4 + (\hat{a}_V^\dagger)^4 + \sqrt{6}(\hat{a}_H^\dagger)^2(\hat{a}_V^\dagger)^2]|0\rangle. \quad (2)$$

First, let the four-photon state be incident to a 50-50 non-polarization beam splitter. For simplicity, we assume that at the beam splitter \hat{a} is transformed into $(1/\sqrt{2})(\hat{b} + \hat{c})$ [10], where \hat{b} and \hat{c} denote the transmitted mode and reflected mode, respectively. By expanding Eq. (2) and keeping only those terms which contain two photons in each of the beams, the state can be written as

$$\{\sqrt{6}[(\hat{b}_H^\dagger)^2(\hat{c}_H^\dagger)^2 + (\hat{b}_V^\dagger)^2(\hat{c}_V^\dagger)^2] + (\hat{b}_H^\dagger)^2(\hat{c}_V^\dagger)^2 + (\hat{b}_V^\dagger)^2(\hat{c}_H^\dagger)^2 + 4\hat{b}_H^\dagger\hat{b}_V^\dagger\hat{c}_H^\dagger\hat{c}_V^\dagger\}|0\rangle. \quad (3)$$

Next, mode \hat{b} and mode \hat{c} are directed to another 50-50 nonpolarization beam splitter. Again, we assume that at the beam splitter \hat{b} is transformed into $(1/\sqrt{2})(\hat{k} + \hat{l})$ and \hat{c} is transformed into $(1/\sqrt{2})(\hat{m} + \hat{n})$, where \hat{k} , \hat{m} , and \hat{l} , \hat{n} denote the transmitted mode and reflected mode, respectively. After this transform, the state can be given by

$$\begin{aligned} & [\sqrt{6}(\hat{k}_H^\dagger\hat{l}_H^\dagger\hat{m}_H^\dagger\hat{n}_H^\dagger + \hat{k}_V^\dagger\hat{l}_V^\dagger\hat{m}_V^\dagger\hat{n}_V^\dagger) \\ & + \hat{k}_H^\dagger\hat{l}_H^\dagger\hat{m}_V^\dagger\hat{n}_V^\dagger + \hat{k}_V^\dagger\hat{l}_V^\dagger\hat{m}_H^\dagger\hat{n}_H^\dagger \\ & + (\hat{k}_H^\dagger\hat{l}_V^\dagger + \hat{k}_V^\dagger\hat{l}_H^\dagger) \times (\hat{m}_H^\dagger\hat{n}_V^\dagger + \hat{m}_V^\dagger\hat{n}_H^\dagger)]|0\rangle. \quad (4) \end{aligned}$$

After the normalization of the four-photon state described by Eq. (4), we get

$$\begin{aligned} |\Psi^{(4)}\rangle = & (\sqrt{3}/3)(|HHHH\rangle_{klmn} + |VVVV\rangle_{klmn}) + (\sqrt{2}/6) \\ & \times (|HHVV\rangle_{klmn} + |VVHH\rangle_{klmn} + |HVHV\rangle_{klmn} \\ & + |VHVH\rangle_{klmn} + |VHHV\rangle_{klmn} + |HVVH\rangle_{klmn}). \quad (5) \end{aligned}$$

For the first term in Eq. (5), the four photons with the same polarization are generated directly by one of the two type-I crystals, whereas the second term comes from two independent photon pairs ($|HH\rangle$ and $|VV\rangle$) simultaneously emitted by each of the two type-I crystals.

In order to investigate the entanglement property of this four-photon entangled state, we analyze at first the correlation between the polarization measurement results. If we use the polarization analysis basis $(1/\sqrt{2})(|V_x\rangle + h e^{-i\phi_x}|H_x\rangle)$, where $\phi_x (x=k, l, m, n)$ is the local phase setting or analysis angle, and $h = \pm 1$ are the two possible measurement results obtained from the two outputs of the polarization beam splitter used by four observers k, l, m , and n , then the correlation function can be given by the following equation [17]:

$$\begin{aligned} E(\phi_k, \phi_l, \phi_m, \phi_n) = & (2/3) \cos(\phi_k + \phi_l + \phi_m + \phi_n) \\ & + (2/9) \cos(\phi_k - \phi_l) \cos(\phi_m - \phi_n) \\ & + (1/9) \cos(\phi_k + \phi_l - \phi_m - \phi_n). \quad (6) \end{aligned}$$

For several specific local phase settings, one can obtain perfect four-photon correlations. For example, the correlation

function is equal to 1 for all local phases equal to zero or $\pi/2$, whereas it is equal to -1 for $\phi_k = \pi$, $\phi_l = \phi_m = \phi_n = 0$.

Based on the correlation function described by Eq. (6), we can investigate a violation of the nonconventional Bell inequality introduced by Ref. [17]. From the local hidden variable theory, the Bell inequality should satisfy $S_L^{(4)} \leq 1$, whereas we can calculate from Eq. (6) that the quantum prediction for the Bell inequality can be as high as $S_{QM}^{(4)} = 8/3\sqrt{2}$ for particular phase settings $\phi_k^{1,2} = 0, \pi/2$, $\phi_l^{1,2} = \pm \pi/4$, $\phi_m^{1,2} = \pm \pi/4$, and $\phi_n^{1,2} = \pm \pi/4$. Therefore, whenever the correlation function implied by this four-photon state has a visibility larger than 0.53, it will result in a violation of the Bell inequality. For ease of operation and high intensity of this four-photon entangled state, this will be a useful tool for testing the local hidden variable theory.

Next we will discuss how this four-photon entangled state can be used in multiparty quantum cryptography, e.g., secret sharing [18]. Suppose party k now wants to send a secret key to parties l, m , and n in such a way that only by working together can they determine what the secret key is. The procedure can be described as follows: First, four parties k, l, m , and n do the polarization measurements on their own photons, party k randomly switches his analysis angles ϕ_k between 0 and $\pi/2$, and parties l, m , and n randomly switch $\phi_x (x=l, m, n)$ between $\pm \pi/4$, with a certain probability to 0. Secondly, after some measurement, parties l, m , and n declare publicly when they have selected the basis of $\pm \pi/4$ and their settings and detected results, at which point party k can calculate the Bell inequality $S^{(4)}$. If there is no eavesdropping, the quantum prediction for $S^{(4)}$ should be $8/3\sqrt{2}$. The existence of any eavesdropping will reduce the entanglement and the violation of the Bell inequality. If the parties find the violation of the Bell inequality is not reduced, then they can assume the remaining cases have been securely transmitted. Party k now declares when he used the analysis angle 0, and then the four parties keep only the measurement results for which they have used the same analysis angle 0.

Now the secure key can be obtained in this way. First we rewrite the four-photon entangled state on the basis of $|X\rangle$ and $|Y\rangle$,

$$\begin{aligned} |\Psi^{(4)}\rangle = & (\sqrt{3}/3)(|XXXX\rangle_{klmn} + |YYYY\rangle_{klmn}) + (\sqrt{2}/6) \\ & \times (|XXYY\rangle_{klmn} + |YYXX\rangle_{klmn} + |XYXY\rangle_{klmn} \\ & + |YXYX\rangle_{klmn} + |YXXY\rangle_{klmn} + |XYYY\rangle_{klmn}). \quad (7) \end{aligned}$$

Here $|X\rangle$ and $|Y\rangle$ correspond to two possible output states of the polarization beam splitters for analysis angle of 0, respectively. They are defined by

$$\begin{aligned} |X\rangle_x = & (1/\sqrt{2})(|H\rangle + |V\rangle), \\ |Y\rangle_x = & (1/\sqrt{2})(|H\rangle - |V\rangle), \\ & x = k, l, m, n. \quad (8) \end{aligned}$$

From Eq. (7), we can see this state has an even number of $|X\rangle$ or $|Y\rangle$. This means if two of the parties m and n agree to

cooperate with party l , all three of them can determine what the result of party k was, i.e., they simply count the number of $|X\rangle$ measurements. If it is even, then party k must have found $|Y\rangle$; if it is odd, then party k must have measured $|X\rangle$. After this, party k asks party l to reveal some of the key. If the bit-error rate is within a reasonable level, then party k can assume that all the remaining three parties have cooperated and he can announce the key is a useful one. Thus a secure key can be established. Let us see how this works in more detail. If only one of the parties m and n (assume it is m) agrees to cooperate with party l , they try to get the measurement results of party k by themselves. If their results are the same (the probability is $7/9$), then the state of the remaining two parties will be projected to

$$\langle FF' | \otimes | \Psi^{(4)} \rangle = (\sqrt{2}/6)(\sqrt{6}|FF\rangle + |F'F'\rangle)_{kn}. \quad (9)$$

Here F and F' are orthogonal to each other, namely $FF' \in (XY, YX)$. Now parties l and m can guess the result of party k (the same as theirs) with a probability of $\frac{6}{7}$. If their measurement results are different (the probability is $2/9$), the state of the remaining two parties can be given by

$$\langle FF' | \otimes | \Psi^{(4)} \rangle = (\sqrt{2}/6)(|FF'\rangle + |F'F'\rangle)_{kn}. \quad (10)$$

In this case, parties l and m can only guess the result of party k with a probability of $\frac{1}{2}$. On average, the probability they will guess right is $\frac{7}{9} \times \frac{6}{7} + \frac{2}{9} \times \frac{1}{2} = \frac{7}{9}$. If both of the parties m and n do not agree with party l , by similar analysis still on average he can guess correctly $7/9$ of the time. After the communication, party k asks party l to declare a subset of the secret key. If the bit-error rate is higher than $2/9$, then party k can conclude the three parties do not come to an agreement, and he can announce that this key should not be used.

In conclusion, a four-photon entangled state directly generated by a two-crystal geometry is proposed. This state can violate the generalized Bell inequality strongly (for local hidden variable theory $S_L^{(4)} \leq 1$, whereas for this state the quantum prediction is $S_{QM}^{(4)} = 8/3\sqrt{2}$). Also, at specific polarization analysis settings, one can obtain perfect correlation, i.e., the value of the correlation function can be 1 or -1 . As an example of an application of this state, we propose that this state can be used in quantum secret sharing. It is expected that this high-intensity four-photon entangled state can also be applied to other fields of quantum information.

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APPENDIX

The process of the parametric down-conversion pumped by a coherent pulse can be approximately described by the following state by dropping higher-order terms [16,19]:

$$|\Psi\rangle = (1 - |\eta|^2/2)|\text{vac}\rangle + \eta|\Phi_1\rangle + \eta^2|\Phi_2\rangle. \quad (A1)$$

Here $|\eta|^2$ is the probability of creating one photon pair in a single pump pulse. $|\Phi_1\rangle$ and $|\Phi_2\rangle$ are a two-photon term and a four-photon term, respectively. They have the following form:

$$\begin{aligned} |\Phi_1\rangle &= \int d\omega_1 d\omega_2 \Phi(\omega_1, \omega_2) \hat{a}_s^\dagger(\omega_1) \hat{a}_i^\dagger(\omega_2) |\text{vac}\rangle, \\ |\Phi_2\rangle &= \int d\omega_1 d\omega_2 d\omega'_1 d\omega'_2 \Phi(\omega_1, \omega_2) \Phi(\omega'_1, \omega'_2) \\ &\quad \times \hat{a}_s^\dagger(\omega_1) \hat{a}_i^\dagger(\omega_2) \hat{a}_s^\dagger(\omega'_1) \hat{a}_i^\dagger(\omega'_2) |\text{vac}\rangle, \end{aligned} \quad (A2)$$

where $\hat{a}_s^\dagger(\omega)$ and $\hat{a}_i^\dagger(\omega)$ are the photon creation operators for the signal and idler modes with frequency ω , respectively. $\Phi(\omega_1, \omega_2)$ is the two-photon wave amplitude in the frequency domain [19,20].

The probability of producing exactly two photons in one single pump pulse is

$$P_2 = |\eta|^2 \langle \Phi_1 | \Phi_1 \rangle = |\eta|^2 \int d\omega_1 d\omega_2 |\Phi(\omega_1, \omega_2)|^2. \quad (A3)$$

The probability of producing exactly four photons in one single pump pulse is

$$\begin{aligned} P_4 &= |\eta|^4 \langle \Phi_2 | \Phi_2 \rangle \\ &= (|\eta|^4/2) \int d\omega_1 d\omega_2 d\omega'_1 d\omega'_2 [|\Phi(\omega_1, \omega_2) \Phi(\omega'_1, \omega'_2)|^2 \\ &\quad + \Phi(\omega_1, \omega_2) \Phi(\omega'_1, \omega'_2) \Phi^*(\omega_1, \omega'_1) \Phi^*(\omega'_2, \omega_2)] \\ &= (P_2^2 + \varepsilon)/2. \end{aligned} \quad (A4)$$

Because of the Schwatz inequality [16,19], $\varepsilon \leq P_2^2$. If an optical filter with bandwidth much narrower than that of $\Phi(\omega_1, \omega_2)$ is used to filter the down-conversion field, then the equality can hold, $\varepsilon = P_2^2$. Under this condition, we have $P_4 = P_2^2$.

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